

PERMUTATIONS OF STRONGLY SELF-ABSORBING C^* -ALGEBRAS

ILAN HIRSHBERG AND WILHELM WINTER

ABSTRACT. Let G be a finite group acting on $\{1, \dots, n\}$. For any C^* -algebra \mathcal{A} , this defines an action of α of G on $\mathcal{A}^{\otimes n}$. We show that if \mathcal{A} tensorially absorbs a UHF algebra of infinite type, the Jiang-Su algebra, or is approximately divisible, then $\mathcal{A} \rtimes_{\alpha} G$ has the corresponding property as well.

1. INTRODUCTION

Recall (see [TW1]) that a separable, unital infinite dimensional C^* -algebra \mathcal{D} is said to be *strongly self-absorbing* if the embedding $\mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{D}$ given by $d \mapsto d \otimes 1$ is approximately unitarily equivalent to an isomorphism (regardless of which tensor product one a-priori uses, any such \mathcal{D} must be nuclear, and thus there is no ambiguity in the definition). The list of known examples of such algebras is quite short. It consists of UHF algebras of ‘infinite type’ (i.e., ones for which all the primes that appear in the supernatural number do so with infinite multiplicity), the Jiang-Su algebra \mathcal{Z} ([JS]), the Cuntz algebras \mathcal{O}_2 and \mathcal{O}_{∞} , and tensor products of \mathcal{O}_{∞} by UHF algebras of infinite type.

A C^* -algebra \mathcal{A} is said to be *\mathcal{D} -absorbing* for a given strongly self-absorbing C^* -algebra \mathcal{D} if $\mathcal{A} \cong \mathcal{A} \otimes \mathcal{D}$. This concept plays an important role in structure theory of C^* -algebras, particularly in relation to the Elliott program.

Various permanence properties of strongly self-absorbing C^* -algebras were studied in [TW1], [HW] and [HRW]. While the property of \mathcal{D} -absorption does remain permanent under many constructions, it does not pass in general to crossed products; one can find counterexamples for $\mathcal{D} = \mathcal{O}_2$ as well as for UHF algebras. However, it was shown in [HW] that \mathcal{D} -absorption does pass to crossed products by \mathbb{Z} , \mathbb{R} or by compact groups provided the group action has a Rokhlin property (see [I1, I2, I3]; for the non-compact cases, one needs to assume that \mathcal{D} is K_1 -injective). We do not know whether there is an example in which \mathcal{Z} -absorption does not pass to crossed products.

The Rokhlin property for finite groups is very restrictive, and it seems desirable to try to look at other examples of group actions. In this note, we consider the following kinds of actions. Suppose G is a finite group acting on $\{1, \dots, n\}$. Let \mathcal{A} be a separable C^* -algebra. We have an induced action $\alpha : G \rightarrow \text{Aut}(\mathcal{A}^{\otimes n})$ given by

$$\alpha_g(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = a_{g(1)} \otimes a_{g(2)} \otimes \cdots \otimes a_{g(n)}$$

(any tensor product can be used here). We note that in many cases of interest, such an action will fail to have the Rokhlin property; for instance, if $\mathcal{A} = \mathcal{Z}$, then $\mathcal{A}^{\otimes n} \cong \mathcal{Z}$ is projectionless, and thus no action on it could have the Rokhlin property

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(in other cases, in which projections do exist, the action typically will not have the Rokhlin property, but might have the weaker *tracial Rokhlin property*, see [P]).

For any C^* -algebra \mathcal{A} , we denote

$$\mathcal{A}_\infty = \ell^\infty(\mathbb{N}, \mathcal{A}) / \mathcal{C}_0(\mathbb{N}, \mathcal{A}).$$

\mathcal{A} may be embedded into $\ell^\infty(\mathbb{N}, \mathcal{A})$ and into \mathcal{A}_∞ in a canonical way (as constant sequences).

If $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ is an action of a finite group G on a C^* -algebra \mathcal{A} , then we have naturally induced actions of G on $\mathcal{M}(\mathcal{A})$, and therefore also on $\ell^\infty(\mathbb{N}, \mathcal{M}(\mathcal{A}))$, $\mathcal{M}(\mathcal{A})_\infty$ and $\mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$, respectively – those actions will all be denoted by $\bar{\alpha}$.

The main result here is the following.

Theorem 1.1. *Let \mathcal{D} be one of the known strongly self-absorbing C^* -algebras. Let α be an action of the finite group G on the separable C^* -algebra \mathcal{A} , and suppose G acts on the set $\{1, \dots, n\}$. Suppose furthermore there exists a unital homomorphism $\psi : \mathcal{D}^{\otimes n} \rightarrow \mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$ which is G -equivariant with respect to the permutation action on $\mathcal{D}^{\otimes n}$ and the induced action $\bar{\alpha}$ on $\mathcal{M}(\mathcal{A})_\infty$. Then, $\mathcal{A} \times_\alpha G$ is \mathcal{D} -absorbing.*

Replacing \mathcal{D} by a finite direct sum of matrix algebras other than \mathbb{C} in the preceding hypotheses, it follows that $\mathcal{A} \times_\alpha G$ is approximately divisible.

In particular, if \mathcal{A} is \mathcal{D} -stable, or \mathcal{A} is approximately divisible, then so is $\mathcal{A}^{\otimes n} \times G$, where G acts on $\mathcal{A}^{\otimes n}$ by permutation.

To prove the theorem, we shall show that for \mathcal{D} a UHF algebra of infinite type, the Jiang-Su algebra, or an algebra of the form $M_p \oplus M_q$ for $p, q > n$, there exists a unital embedding of \mathcal{D} into the fixed point subalgebra of $\mathcal{D}^{\otimes n}$ under the action of the symmetric group S_n .

For $\mathcal{D} = \mathcal{O}_2$ or \mathcal{O}_∞ , this is already known (see [I1, Theorems 4.2 and Section 5] and [KK, Lemma 10] – for \mathcal{O}_2 , one can show that the action of S_n has the Rokhlin property, and for \mathcal{O}_∞ , this follows from the much more general fact that the fixed point subalgebra of \mathcal{O}_∞ under any finite group action is a finite direct sum of simple purely infinite C^* -algebras).

We wish to point out that the arguments we have use the actual structure of the specific strongly self-absorbing C^* -algebras. It remains an open problem to find a proof that works for any strongly self-absorbing C^* -algebra.

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2. PRELIMINARIES

If $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ is an action, we shall denote by \mathcal{A}^α , or by \mathcal{A}^G , the fixed point subalgebra (we'll use the latter when it is clear what the action is).

We have the following characterization of \mathcal{D} -absorption (based on ideas of Elliott and of Kirchberg), which appears as Theorem 7.2.2 in [R1]. We note that the statement in [R1] refers to the relative commutant of \mathcal{A} in an ultrapower of $\mathcal{M}(\mathcal{A})$; however, it is easy to see that the characterization still holds as stated below.

Theorem 2.1. *Let \mathcal{D} be a strongly self-absorbing and \mathcal{A} be any separable C^* -algebra. Then, \mathcal{A} is \mathcal{D} -absorbing if and only if \mathcal{D} admits a unital homomorphism to $\mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$.*

Note that since any strongly self-absorbing C^* -algebra has to be simple, it follows that unless $\mathcal{A} = 0$, the $*$ -homomorphism above must be an embedding (of course, 0 is \mathcal{D} -absorbing for any \mathcal{D}).

We note that in the above theorem, if instead of strongly self-absorbing we take $\mathcal{D} = M_p \oplus M_q$ for some $p, q > 1$, then \mathcal{D} -absorption should be replaced by approximate divisibility. It is furthermore easy to show that if $\mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$ admits a unital homomorphism from M_p , then it also admits a unital homomorphism from the UHF algebra M_{p^∞} .

A simple modification of Lemma 2.3 from [HW] gives us the following.

Lemma 2.2. *Let \mathcal{A}, \mathcal{D} be unital separable C^* -algebras. Let G be a finite group and $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ be an action. Suppose \mathcal{D} admits a unital homomorphism into $(\mathcal{A}_\infty \cap \mathcal{A}')^{\bar{\alpha}}$. Then, \mathcal{D} admits a unital homomorphism into $(\mathcal{M}(\mathcal{A} \rtimes_\alpha G))_\infty \cap (\mathcal{A} \rtimes_\alpha G)'$.*

In particular, it follows that if $\mathcal{D} = M_p$ above, then $\mathcal{A} \rtimes_\alpha G$ absorbs M_{p^∞} . If $\mathcal{D} = M_p \oplus M_q$ for some $p, q > 1$, then $\mathcal{A} \rtimes_\alpha G$ is approximately divisible.

Remark 2.3. In the situation of Theorem 1.1, note that, since ψ is G -equivariant, the S_n -invariant subalgebra of the copy of $\mathcal{D}^{\otimes n}$ in $\mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$ is clearly contained in the fixed point subalgebra of $\mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$ under the action of G . It will thus suffice for us to show that there is a unital embedding of \mathcal{D} into $(\mathcal{D}^{\otimes n})^{S_n}$.

Henceforth, therefore, we assume that $G = S_n$, and that α is given by permutation of the tensor factors.

We shall also require a result from [HRW] concerning $C(X)$ -algebras. Recall that, for a compact Hausdorff space X , a $C(X)$ -algebra is a C^* -algebra \mathcal{A} , along with a fixed unital homomorphism from $C(X)$ to the center of $\mathcal{M}(\mathcal{A})$. The fiber over $x \in X$, denoted \mathcal{A}_x , is the quotient $\mathcal{A}/C_0(X \setminus \{x\})\mathcal{A}$.

Theorem 2.4 ([HRW], Theorem 4.6). *Let \mathcal{D} be a K_1 -injective strongly self-absorbing C^* -algebra. Let X be a compact metrizable space of finite covering dimension. Let \mathcal{A} be a separable $C(X)$ -algebra. It follows that \mathcal{A} is \mathcal{D} -absorbing if and only if \mathcal{A}_x is \mathcal{D} -absorbing for each $x \in X$.*

3. PROOF OF THE MAIN THEOREM

Lemma 3.1. *For some $m, n \in \mathbb{N}$, consider the action α of S_n on $M_m^{\otimes n}$ given by permutation of the factors. Suppose there are $p, k \in \mathbb{N}$ are such that p is prime, $p^k | m$ and $p^k \nmid n!$. Then, there exists a unital embedding of M_p into $(M_m^{\otimes n})^{S_n}$.*

Proof. Denote $V = \mathbb{C}^m$. Note that $M_m^{\otimes n} \cong \mathcal{B}(V^{\otimes n})$. We have a unitary representation U of S_n on $V^{\otimes n}$ given by

$$U_g(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \xi_{g(1)} \otimes \xi_{g(2)} \otimes \cdots \otimes \xi_{g(n)}$$

Note that $\alpha_g(a) = U_g a U_g^*$. Thus, the fixed point subalgebra is

$$\{U_g \mid g \in S_n\}' \cong \bigoplus_{\rho \in \hat{S}_n} M_{\mu(\rho)}$$

where \hat{S}_n is the set of (equivalence classes of) irreducible representations of S_n , and $\mu(\rho)$ is the multiplicity of the representation ρ as a subrepresentation of U . It suffices, therefore, to show that $p | \mu(\rho)$ for all $\rho \in \hat{S}_n$. To this effect, we shall compute the character χ of the representation U .

Suppose $g \in S_n$ is represented as ℓ disjoint cycles (including cycles of length 1). Let ξ_1, \dots, ξ_m be an orthonormal basis for V . So, $\{\xi_{i_1} \otimes \dots \otimes \xi_{i_n} \mid i_1, \dots, i_n \in \{1, \dots, m\}\}$ form an orthonormal basis for $V^{\otimes n}$. So, we have

$$\begin{aligned} \chi(g) &= \sum_{i_1, \dots, i_n \in \{1, \dots, m\}} \langle \xi_{i_1} \otimes \dots \otimes \xi_{i_n}, \xi_{g(i_1)} \otimes \dots \otimes \xi_{g(i_n)} \rangle \\ &= \sum_{i_1, \dots, i_n \in \{1, \dots, m\}} \delta_{(i_1, \dots, i_n), (g(i_1), \dots, g(i_n))}, \end{aligned}$$

where the last expression is simply the number of n -tuples of elements from $\{1, \dots, m\}$ which are fixed under the permutation g . An n -tuple is fixed if and only if it is constant on each of the cycles of g , and there are ℓ of those. Thus

$$\chi(g) = m^\ell.$$

Recall that the characters of the irreducible representations of S_n are all integer-valued (this follows immediately from the Frobenius character formula; see, for instance, chapter 4 of [FH]). Thus, suppose ρ is an irreducible representation, and its character is χ_ρ , then

$$\mu(\rho) = \frac{1}{n!} \langle \chi_\rho, \chi \rangle$$

Since all the entries of χ are integers divisible by m , and hence by p^k , and χ_ρ consists of integer entries, we see that $\langle \chi_\rho, \chi \rangle$ is an integer and $p^k \mid \langle \chi_\rho, \chi \rangle$, and since $p^k \nmid n!$, we have that $p \mid \mu(\rho)$, as required. \square

The following is now immediate.

Corollary 3.2. *If \mathcal{D} is a strongly self absorbing UHF algebra, and p is a prime which appears in the supernatural number associated to \mathcal{D} , then there is a unital embedding of M_{p^∞} into the fixed point subalgebra of $\mathcal{D}^{\otimes n}$ under the action of S_n .*

Proof. The preceding lemma clearly yields an embedding of M_p into the fixed point algebra of $\mathcal{D}^{\otimes n}$, hence an embedding of M_{p^∞} into the fixed point algebra of $(\mathcal{D}^{\otimes \infty})^n$. Identifying \mathcal{D} with $\mathcal{D}^{\otimes \infty}$, we obtain the result. \square

In view of Remark 2.3 and Lemma 2.2, we thus proved Theorem 1.1 for the case of UHF algebras of infinite type.

For \mathcal{D} the tensor product of \mathcal{O}_∞ with a UHF algebra \mathcal{B} of infinite type, we may apply Theorem 1.1 for \mathcal{O}_∞ and \mathcal{B} separately, using $\psi \circ (\text{id}_{\mathcal{O}_\infty} \otimes \mathbf{1}_{\mathcal{B}})^{\otimes n}$ and $\psi \circ (\mathbf{1}_{\mathcal{O}_\infty} \otimes \text{id}_{\mathcal{B}})^{\otimes n}$, respectively, in place of ψ to show that $\mathcal{A} \times_\alpha G$ absorbs both \mathcal{O}_∞ and \mathcal{B} . But then it is straightforward to conclude from Theorem 2.1 that $\mathcal{A} \times_\alpha G$ also absorbs $\mathcal{O}_\infty \otimes \mathcal{B}$.

We now turn to the case of approximate divisibility. We first need a simple technical lemma. The proof is done via a standard diagonalization trick, which we leave to the reader. Any tensor product can be taken in the lemma below.

Lemma 3.3. *Let \mathcal{A}, \mathcal{D} be C^* -algebras, and let $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ be an action of a finite group. Suppose \mathcal{D} is separable, and let \mathcal{C} be a separable subspace of $\mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$. Suppose there exists a unital homomorphism*

$$\psi : \mathcal{D}^{\otimes n} \rightarrow \mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$$

such that

$$\bar{\alpha}_g(\psi(d_1 \otimes d_2 \otimes \dots \otimes d_n)) = \psi(d_{g(1)} \otimes d_{g(2)} \otimes \dots \otimes d_{g(n)})$$

for all $g \in G$ and d_1, \dots, d_n in \mathcal{D} . Then, there exists a unital homomorphism

$$\psi' : \mathcal{D}^{\otimes n} \rightarrow \mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}' \cap \mathcal{C}'$$

which satisfies the same property with respect to the action $\bar{\alpha}$.

Corollary 3.4. *Under the hypotheses of Theorem 1.1, for \mathcal{D} a finite direct sum of matrix algebras other than \mathbb{C} , we have that $\mathcal{A} \times_\alpha G$ is approximately divisible.*

Proof. Choose two different prime numbers p, q greater than n . By a repeated application of Lemma 3.3, we see that for any m , we can find unital homomorphisms $\psi_1, \dots, \psi_m : \mathcal{D}^{\otimes n} \rightarrow \mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$ with commuting ranges such that

$$\bar{\alpha}_g(\psi_k(d_1 \otimes d_2 \otimes \dots \otimes d_n)) = \psi_k(d_{g(1)} \otimes d_{g(2)} \otimes \dots \otimes d_{g(n)})$$

for all $k = 1, \dots, m$, $g \in G$ and d_1, \dots, d_n in \mathcal{D} . Define $\psi : (\mathcal{D}^{\otimes m})^{\otimes n} \rightarrow \mathcal{M}(\mathcal{A})_\infty \cap \mathcal{A}'$ by $\psi = \psi_1 \otimes \psi_2 \otimes \dots \otimes \psi_m$, then we have

$$\bar{\alpha}_g(\psi(d_1 \otimes d_2 \otimes \dots \otimes d_n)) = \psi(d_{g(1)} \otimes d_{g(2)} \otimes \dots \otimes d_{g(n)})$$

for all $g \in G$ and d_1, \dots, d_n in $\mathcal{D}^{\otimes m}$. Of course, if \mathcal{D}_0 is any unital subalgebra of $\mathcal{D}^{\otimes m}$, we could restrict ψ to \mathcal{D}_0 to get an embedding with similar properties, using \mathcal{D}_0 instead of \mathcal{D} .

We recall that there is a natural number N such that for any natural number $k \geq N$ there are positive integers a, b such that $ap + bq = k$, and therefore there is a unital embedding of $M_p \oplus M_q$ into M_k . Now, if $\ell > 1$ is the size of the smallest matrix algebra summand in \mathcal{D} , the size of the smallest matrix algebra summand in $\mathcal{D}^{\otimes m}$ is ℓ^m , and thus, for sufficiently large m , there exists a unital embedding of $M_p \oplus M_q$ into $\mathcal{D}^{\otimes m}$. We may therefore assume that we had $\mathcal{D} \cong M_p \oplus M_q$ to begin with.

We now show that there is a unital embedding of $M_p \oplus M_q$ into $((M_p \oplus M_q)^{\otimes n})^{S_n}$. Note that we can obviously identify

$$(M_p \oplus M_q)^{\otimes n} \cong \bigoplus_{v \in \{0,1\}^n} \bigotimes_{i=1}^n M_{p^{v(i)}} \otimes M_{q^{1-v(i)}}$$

(where we write elements of $\{0, 1\}^n$ as functions from $\{1, 2, \dots, n\}$ to $\{0, 1\}$). Consider the algebra

$$\mathcal{B} = \bigoplus_{k=0}^n \left(M_p^{\otimes n-k} \right)^{S_{n-k}} \otimes \left(M_q^{\otimes k} \right)^{S_k}$$

where the superscript denotes that we are considering the fixed point subalgebra under the corresponding action of the symmetric subgroup of S_n . For $b \in \mathcal{B}$, we denote by $b(k)$ the component of b in the summand

$$\left(M_p^{\otimes n-k} \right)^{S_{n-k}} \otimes \left(M_q^{\otimes k} \right)^{S_k} \subseteq M_p^{\otimes n-k} \otimes M_q^{\otimes k}.$$

We have a unital embedding of \mathcal{B} into $\bigoplus_{v \in \{0,1\}^n} \bigotimes_{i=1}^n M_{p^{1-v(i)}} \otimes M_{q^{v(i)}}$ defined as follows. For

$$a \in \bigoplus_{v \in \{0,1\}^n} \bigotimes_{i=1}^n M_{p^{1-v(i)}} \otimes M_{q^{v(i)}}$$

we denote by $a(v)$ the v 'th component of $a \in (M_p \oplus M_q)^{\otimes n}$, $v \in \{0, 1\}^n$. Write $k_v = \sum_{i=1}^n v(i)$, and let $w_v \in \{0, 1\}^n$ be given by

$$w_v(i) = \begin{cases} 0 & | \ i \leq n - k \\ 1 & | \ i > n - k \end{cases}$$

Let $g \in S_n$ be a permutation such that $g(w_v) = v$, where we consider the naturally induced permutation action of S_n on $\{0, 1\}^{\otimes n}$. We define

$$\beta_g : M_p^{\otimes n-k_v} \otimes M_q^{\otimes k_v} \rightarrow \bigotimes_{i=1}^n M_{p^{1-v(i)}} \otimes M_{q^{v(i)}}$$

by

$$\beta_g(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = a_{g(1)} \otimes \cdots \otimes a_{g(n)}$$

We note that if $x \in (M_p^{\otimes n-k})^{S_{n-k}} \otimes (M_q^{\otimes k})^{S_k} \subseteq M_p^{\otimes n-k_v} \otimes M_q^{\otimes k_v}$ and $g(w_v) = h(w_v) = v$ then $\beta_g(x) = \beta_h(x)$.

We can define

$$\psi : \mathcal{B} \rightarrow \bigoplus_{v \in \{0,1\}^n} \bigotimes_{i=1}^n M_{p^{1-v(i)}} \otimes M_{q^{v(i)}}$$

by

$$\psi(b)(v) = \beta_{g_v}(b(k_v))$$

where g_v is an element of S_n which satisfies that $g_v(w_v) = v$. In particular, if $v = w_v$, then we simply have $\psi(b)(v) = b(k_v)$.

For any $g \in S_n$, we have

$$\alpha_g(\psi(b))(v) = \beta_{h_1} \beta_{h_2}^{-1}(\psi(b)(g^{-1}(v)))$$

where h_1, h_2 are such that $h_2(g^{-1}(v)) = w_{g^{-1}(v)} = w_v$, and $h_1(w_v) = v$. Note further that $\psi(b)(g^{-1}(v)) = \beta_{h_2} b(k_{g^{-1}(v)})$ and that $k_{g^{-1}(v)} = k_v$, and thus

$$\alpha_g(\psi(b))(v) = \beta_{h_1} \beta_{h_2}^{-1}(\beta_{h_2} b(k_v)) = \beta_{h_1}(b(k_v)) = \psi(b)(v),$$

so indeed the image of ψ is fixed under α_g for all $g \in G$.

It thus suffices to construct a unital embedding $\varphi : M_p \oplus M_q \rightarrow \mathcal{B}$. For this, it will suffice, for each $k = 0, 1, \dots, n$, to construct a unital homomorphism

$$\varphi_k : M_p \oplus M_q \rightarrow (M_p^{\otimes n-k})^{S_{n-k}} \otimes (M_q^{\otimes k})^{S_k}.$$

By Lemma 3.1, for $k < n$, we have a unital embedding of M_p into $(M_p^{\otimes n-k})^{S_{n-k}} \otimes \mathbb{C}1 \subseteq (M_p^{\otimes n-k})^{S_{n-k}} \otimes (M_q^{\otimes k})^{S_k}$, and thus we may select φ_k to be such a non-injective unital homomorphism, which annihilates the summand M_q . For $n = k$, we have that $(M_p^{\otimes n-k})^{S_{n-k}} \otimes (M_q^{\otimes k})^{S_k} \cong (M_q^{\otimes n})^{S_n}$, and again by Lemma 3.1 we can choose a unital embedding of M_q into this summand, and by annihilating the summand M_p , we have a unital homomorphism from $M_p \oplus M_q$. Put together, we get a unital homomorphism $\varphi = \bigoplus_{k=0}^n \varphi_k$, which has a trivial kernel. \square

We now turn to the proof for $\mathcal{D} = \mathcal{Z}$.

Proof of Theorem 1.1 for the case $\mathcal{D} = \mathcal{Z}$. We denote

$$\mathcal{E} = \{f \in C([0, 1], M_{2\infty} \otimes M_{3\infty}) \mid f(0) \in M_{2\infty} \otimes 1, f(1) \in 1 \otimes M_{3\infty}\}$$

By Proposition 2.2 from [R2], we know that one can embed \mathcal{E} unitaly into \mathcal{Z} . Thus, it suffices for us to construct a unital homomorphism from \mathcal{Z} to $(\mathcal{E}^{\otimes n})^{S_n}$.

\mathcal{E} can naturally be regarded as a $C([0, 1])$ -algebra (the center of \mathcal{E} can be identified in the obvious way with $C([0, 1])$), where the fiber \mathcal{E}_0 is $M_{2\infty}$, the fiber \mathcal{E}_1 is $M_{3\infty}$, and the fibers \mathcal{E}_t for $0 < t < 1$ are $M_{2\infty} \otimes M_{3\infty}$. We may thus regard $\mathcal{E}^{\otimes n}$ as a $C([0, 1]^n)$ -algebra, where the fiber over $\vec{t} = (t_1, t_2, \dots, t_n)$ is isomorphic to $\mathcal{E}_{t_1} \otimes \mathcal{E}_{t_2} \otimes \dots \otimes \mathcal{E}_{t_n}$ (see Proposition 1.6 of [HRW] for a discussion of these matters). We shall denote $\mathcal{E}_{\vec{t}} = \mathcal{E}_{t_1} \otimes \mathcal{E}_{t_2} \otimes \dots \otimes \mathcal{E}_{t_n}$.

The unital inclusion

$$\mathcal{E}^{\otimes n} \supseteq (\mathcal{E}^{\otimes n})^{S_n} \supseteq (C([0, 1])^{\otimes n})^{S_n} \cong C([0, 1]^n / S_n) \cong C(\Delta)$$

gives $(\mathcal{E}^{\otimes n})^{S_n}$ and $\mathcal{E}^{\otimes n}$ the structure of $C(\Delta)$ -algebras, where

$$\Delta = \{(t_1, t_2, \dots, t_n) \in [0, 1]^n \mid t_1 \leq t_2 \leq \dots \leq t_n\}.$$

Let $\vec{t} \in \Delta$ be a point, and let H be the isotropy group of \vec{t} . Note that a function $f \in C_0(\Delta \setminus \{\vec{t}\})$, thought of as a function on $[0, 1]^n$, is a function which vanishes on the S_n -orbit of \vec{t} (i.e. on $|G/H|$ points), and hence, thought of as a $C(\Delta)$ -algebra, we have $\mathcal{E}_{\vec{t}}^{\otimes n} \cong \bigoplus_{\vec{s} \in S_n(\vec{t})} \mathcal{E}_{\vec{s}}$. As the S_n action drops to this quotient, we have that $(\mathcal{E}^{\otimes n})_{\vec{t}}^{S_n} \subseteq \mathcal{E}_{\vec{t}}^{\otimes n} \cong \bigoplus_{\vec{s} \in S_n(\vec{t})} \mathcal{E}_{\vec{s}}$ and consists of the S_n -invariants elements there. Each S_n -invariant element in $\bigoplus_{\vec{s} \in S_n(\vec{t})} \mathcal{E}_{\vec{s}}$ is determined by its summand in $\mathcal{E}_{\vec{t}}$, and thus $(\mathcal{E}^{\otimes n})_{\vec{t}}^{S_n}$ is isomorphic to $\mathcal{E}_{\vec{t}}^H$.

Denoting $\{s_1, \dots, s_\ell\} = \{t_1, \dots, t_n\}$, where $s_1 < s_2 < \dots < s_\ell$ and each s_j appears k_j times in the ordered n -tuple (t_1, \dots, t_n) , we have that H is a direct product of the symmetric groups S_{k_j} , $j = 1, \dots, \ell$, and

$$\mathcal{E}_{\vec{t}}^H \cong \bigotimes_{j=1}^{\ell} \left(\mathcal{E}_{s_j}^{\otimes k_j} \right)^{S_{k_j}}.$$

It follows from Corollary 3.2 that $\left(\mathcal{E}_{s_j}^{\otimes k_j} \right)^{S_{k_j}}$ absorbs a UHF algebra (of type 2^∞ or 3^∞), and therefore, so does $\bigotimes_{j=1}^{\ell} \left(\mathcal{E}_{s_j}^{\otimes k_j} \right)^{S_{k_j}}$. By [JS, Theorem 5], any infinite dimensional UHF algebra is \mathcal{Z} -absorbing, and therefore, $\mathcal{E}_{\vec{t}}^H$ is \mathcal{Z} -absorbing as well.

We thus see that all the fibers of the $C(\Delta)$ -algebra $(\mathcal{E}^{\otimes n})^{S_n}$ are \mathcal{Z} -absorbing. From Theorem 2.4, since Δ has finite covering dimension, we see that $(\mathcal{E}^{\otimes n})^{S_n}$ must be \mathcal{Z} -absorbing, and in particular, admits a unital embedding of \mathcal{Z} . Therefore, there is a unital embedding of \mathcal{Z} into $(\mathcal{Z}^{\otimes n})^{S_n}$, as required. \square

REFERENCES

- [FH] W. Fulton and J. Harris, Representation theory. A first course. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.
- [HRW] I. Hirshberg, M. Rørdam, W. Winter, $C_0(X)$ -algebras, stability and strongly self-absorbing C^* -algebras. Preprint (2006), ArXiv:math.OA/0610344; to appear in Math. Ann.
- [HW] I. Hirshberg, W. Winter, Rokhlin actions and self-absorbing C^* -algebras. Preprint (2005), ArXiv:math.OA/0505356; to appear in Pacific J. Math.

- [I1] M. Izumi, Finite group actions on C^* -algebras with the Rohlin property I. *Duke Math. J.* 122 (2004), no. 2, 233–280.
- [I2] M. Izumi, Finite group actions on C^* -algebras with the Rohlin property II. *Adv. Math.* 184 (2004), no. 1, 119–160.
- [I3] M. Izumi, The Rohlin property for automorphisms of C^* -algebras. *Mathematical physics in mathematics and physics* (Siena, 2000), 191–206, *Fields Inst. Commun.*, 30, Amer. Math. Soc., Providence, RI, 2001.
- [JS] X. Jiang, H. Su, On a simple unital projectionless C^* -algebra. *Amer. J. Math.* 121 (1999), no. 2, 359–413.
- [KK] A. Kishimoto and A. Kumjian, Crossed products of Cuntz algebras by quasi-free automorphisms. *Operator Algebras and Their Applications* (Waterloo, Ontario, Canada, 1994/1995), 173–192, *Fields Inst. Commun.* 13, Amer. Math. Soc., Providence, RI, 1997.
- [P] N.C. Phillips, The tracial Rokhlin property for actions of finite groups on C^* -algebras. Preprint, ArXiv:math.OA/0609782.
- [R1] M. Rørdam, Classification of nuclear, simple C^* -algebras, *Encyclopaedia Math. Sci.*, 126, Springer, Berlin, 2002.
- [R2] M. Rørdam, The stable and the real rank of \mathcal{Z} -absorbing C^* -algebras. *Internat. J. Math.* 15 (2004), no. 10, 1065–1084.
- [TW1] A. Toms, W. Winter, Strongly self-absorbing C^* -algebras. *Trans. Am. Math. Soc.* 359 (2007), 3999–4029.
- [TW2] A. Toms, W. Winter, \mathcal{Z} -stable ASH algebras. Preprint, ArXiv:math.OA/0508218; to appear in *Can. J. Math.*

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE’ER SHEVA 84105, ISRAEL

E-mail address: ilan@math.bgu.ac.il

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNSTER, EINSTEINSTR. 62, 48149 MÜNSTER, GERMANY

E-mail address: wwinter@math.uni-muenster.de